Binary Stars - astro8501 - 6944

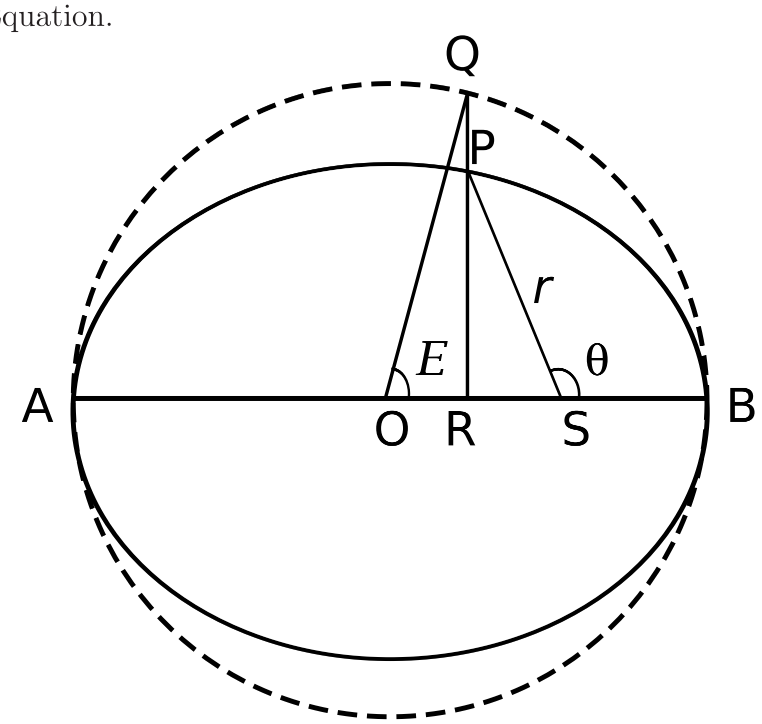
Problem Sheet 2

1. Our solar system is not a binary star but it contains the planet Jupiter. Calculate the orbital angular momentum stored in Jupiter and the spin angular momentum of the Sun. Where is most of the angular momentum of the solar system? Do the other planets store significant angular momentum? Are there implications for formation of stars and solar systems?

2. Given that the velocity vector $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$, where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors in the radial and transverse directions of motion, find an expression for the acceleration $\ddot{\mathbf{r}}$. The angular momentum vector is $\mathbf{J} = m\mathrm{r} \times \dot{\mathbf{r}}$. Find its magnitude J. Given Kepler's second law, show $J/2m$ is a constant and that the transverse acceleration term is zero (and say physically why this should be so).

3. Given the orbital motion equation $r = l/(1 + e\cos\theta)$, where $l = a(1 - e^2) = h^2/G(M_1 + M_2)$ is the semi-latus rectum and h the specific angular momentum, find the magntiudes of the radius vector at periastron and apastron. Show that the orbital speed is given by $v_{\rm orb}^2 = |\dot{\mathbf{r}}|^2 = G(M_1 + M_2)\left(\frac{2}{r} - \frac{1}{a}\right)$.
   If β is the angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$, show that $\sin^2\beta = \frac{a^2(1-e^2)}{r(2a-r)}$ hence derive an expression for $\cos\beta$.

4. Kepler's Equation.



The figure shows an elliptical orbit (solid line) with position P given by r, θ. The *eccentric anomaly*, E, is also shown. It connects the stellar position to a position on the *auxiliary circle* (dashed line). Label the ellipse semi-major and semi-major axes, a and b respectively, on your diagram. Show that $r = a(1 - e\cos E)$ and find expressions for $\sin\theta$ and $\cos\theta$ as functions of a, b, e and E only. Hence by calculating $d\theta/dE$ derive Kepler's equation $E - e\sin E = ct + d$ where t is time and c, d are constants. (It is a property of auxiliary circles that $PR/QR = b/a$ - there are bonus points if you can show this simply!) How would you go about solving Kepler's equation?

5. The generalised Runge-Lenz vector is defined as $\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}$ where m is the particle mass moving under a central force $\mathbf{F}(\mathbf{r}) = -k\hat{\mathbf{r}}r^{-2}$, $\mathbf{p}$ its momentum vector, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ its angular momentum vector, k describes the strength of the central force, $\mathbf{r}$ is the position of the particle and $\hat{\mathbf{r}} = \mathbf{r}/r$.
   Write down an expression for the energy E in terms of the above vectors.
   By finding $\mathbf{A} \cdot \mathbf{r}$ show the motion is in a conic section.
   For a generalised central force $\mathbf{F} = d\mathbf{p}/dt = f(r)\hat{\mathbf{r}}$, show that
   $$\frac{d\,(\mathbf{p} \times \mathbf{L})}{dt} = -mf(r)r^2 \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$$
   hence that $\mathbf{A}$ is conserved for an inverse-square force field.

6. The Doppler effect on wavelength of spectral lines is given by
   $$\begin{aligned} \frac{\lambda - \lambda_0}{\lambda} &= \frac{\Delta\lambda}{\lambda} \\ &= \left(\frac{1 + V/c}{1 - V/c}\right)^{0.5} - 1\,. \end{aligned}$$
   Explain the meaning of the symbols and show that $\Delta\lambda \approx V\lambda/c$ for small V.
   Hence calculate the Doppler shift corresponding to $V = 50\,\mathrm{km\,s^{-1}}$ and $\lambda = 4500\,$.
   What resolution is required to observe the effect?

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