Eddy viscosity and turnover timescale

Consider the convective timescale $\tau_{\rm conv}$ on which the largest convective cells of length l turnover. The mass in the cells is

$$M = \rho V = \rho\,\sigma\,l \tag{1}$$

for cell cross-section σ and fluid density ρ. Such a cell exchanges material separated by a distance l. If there is a velocity gradient in the medium such that one end of the cell is moving at u and the other at $u + du$, then the cell transports linear momentum

$$\Delta p = M\,\Delta u = \rho\,\sigma\,l\,du \tag{2}$$

but

$$\Delta p = F\,\Delta t = F\,l/v_{\rm conv} \tag{3}$$

and force per unit area is

$$F/\sigma = \rho\,\nu\,du/dz = \rho\,\nu\,du/l \tag{4}$$

where ν is the viscosity of the fluid. Thus

$$\frac{v_{\rm conv}}{l}\frac{\rho\,\sigma\,l\,du}{\sigma} = \rho\,\nu\frac{du}{l} \tag{5}$$

$$\Rightarrow \quad \nu = l\,v_{\rm conv} \tag{6}$$

or, averaged over three dimensions,

$$\nu = \frac{1}{3}l\,v_{\rm conv} = \frac{1}{3}l^2/\tau_{\rm conv}\,. \tag{7}$$

Following Schwarzschild (1958) the average energy flux carried by convective motions can be written as

$$H = \nabla\Delta T\,dr\,C_p\rho\,v_{\rm conv} \tag{8}$$

if dr is the average convective displacement and $v_{\rm conv}$ is the average velocity of all convective elements at level r. The quantity

$$\nabla\Delta T = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} - \frac{dT}{dr} \tag{9}$$

represents the temperature excess of the actual temperature gradient over the adiabatic temperature gradient. This is the temperature excess of the convective element after it has risen by a distance dr. The element rises due to an average excess force acting on it, which can be multiplied by the distance dr to obtain the work done on the element, which in turn produces the kinetic energy of the element

$$\frac{1}{2}\rho\, v_{\rm conv}^2 = \frac{\rho}{T}\,\nabla\Delta T\, dr \frac{GM_r}{r^2}\frac{1}{2}dr \tag{10}$$

which gives the convective velocity as a function of the temperature gradient. Therefore

$$H = \rho\, C_p \left(\frac{GM_r}{Tr^2}\right)^{1/2} (\nabla\Delta T)^{3/2}\,\frac{l^2}{4} \tag{11}$$

where M_r is the mass inside a shell of radius r and $\overline{dr} = \frac{1}{2}l$ has been used. From the virial theorem

$$E = \frac{1}{2}E_G \tag{12}$$

$$\frac{GM_r}{2r} \simeq C_p T\,. \tag{13}$$

Putting all this together gives

$$H = \frac{v_{\rm conv}^2\, r}{2dr^2 C_p} dr\, C_p\, \rho\, v_{\rm conv} = \frac{\rho\, r\, v_{\rm conv}^3}{2\, dr}\,. \tag{14}$$

The luminosity is

$$L_r = 4\pi r^2 H \tag{15}$$

so that,

$$\frac{L_r}{4\pi r^2} \simeq \frac{\rho_r v_{\rm conv}^3}{l}\,. \tag{16}$$

Now

$$\rho_r = \frac{M_r}{V_r} \simeq \frac{M_{\rm env}}{4\pi r^2 l} \tag{17}$$

$$\Rightarrow \quad v_{\rm conv}^3 = \frac{L_r\, l^2}{M_{\rm env} r}\,, \tag{18}$$

thus,

$$\tau_{\rm conv} = \frac{l}{v_{\rm conv}} = \left(\frac{M_{\rm env}\, r\, l}{L_r}\right)^{1/3}\,. \tag{19}$$

Taking

$$l \simeq \frac{1}{3}R_{\rm env} \tag{20}$$

and

$$r \simeq \left(R - \frac{1}{2}R_{\rm env}\right) \tag{21}$$

as the radius in the middle of the convective zone, gives

$$\tau_{\rm conv} \simeq \left(\frac{M_{\rm env}R_{\rm env}\left(R - \frac{1}{2}R_{\rm env}\right)}{3L}\right)^{1/3}\,. \tag{22}$$