The Henyey Scheme

1 A Star in Shells

Stars are usually modelled as a series of spherical shells, labelled by the Lagrangian mass co-ordinate m. There are usually four stellar structure equations and hence four independent variables.

The equations are hydrostatic equilibrium

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}, \tag{1}$$

mass conservation

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}, \tag{2}$$

nuclear energy generation

$$\frac{dL}{dm} = \epsilon(X, T, \rho) \tag{3}$$

and radiative transport of the energy flux $\mathcal{F}$,

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa}{T^3}\frac{\mathcal{F}}{\left(4\pi r^2\right)^2}. \tag{4}$$

The independent variables are usually radius r, luminosity L, temperature T and one other, commonly P, which is related to the degeneracy, composition and density ρ through the equation of state.

Note that the number of equations, n, is typically four but is larger if, e.g., rotation and composition are also solved for, also perhaps there is an equation for velocity u (if hydrostatic equilibrium is not assumed).

Define variables:

- Number of equations n
- Number of shells N
- Equation number i (from 1 to n)
- Shell number j (from 1 to N)

- Structure variables

$$\begin{aligned} \mathbf{x} &= x_k = \left\{x_1, x_2, x_3, x_4\right\} \\ &= \{L, r, \rho, T\} \end{aligned} \tag{5}$$

- The corrections are

$$\begin{aligned} \delta\,\mathbf{x} &= \delta x_k = \delta\left\{x_1, x_2, x_3, x_4\right\} \\ &= \delta\{L, r, \rho, T\} \end{aligned} \tag{6}$$

There are then N shells, and n equations at each shell, for a total of $N \times n$ equations to be solved for a star.

2 Setup of the equations

The equations are written in a form such that the right hand side is always zero when they are solved perfectly, e.g. for the hydrostatic equation at shell j

$$\frac{dP_j}{dm_j} + \frac{Gm_j}{4\pi r_j^4} = 0\,. \tag{7}$$

In general the equation is not solved exactly: we have P and r from the previous timestep (or some extrapolation thereof) and at the next timestep there will be a residual such that

$$\frac{dP_j}{dm_j} + \frac{Gm_j}{4\pi r_j^4} = -g_j \neq 0. \tag{8}$$

Corrections δP_j and δr_j are applied to P_j and r_j such that the equation is solved exactly (to within some tolerance),

$$\frac{d(P_j + \delta P_j)}{dm_j} + \frac{Gm_j}{4\pi(r_j + \delta r_j)^4} \;=\; 0\,. \tag{9}$$

The aim is then to determine the δP_j and δr_j : of course it is not trivial because there are four coupled equations at each shell which must be solved for, and then the shells are also coupled.

3 Taylor series expansion

We can Taylor expand the equation around the previous solution,

$$
\begin{aligned}
-g_j \quad = \quad & \frac{\partial g_j}{\partial P_{j-1}}\delta P_{j-1} + \frac{\partial g_j}{\partial r_{j-1}}\delta r_{j-1} + \\
& \frac{\partial g_j}{\partial P_j}\delta P_j + \frac{\partial g_j}{\partial r_j}\delta r_j + \\
& \frac{\partial g_j}{\partial P_{j+1}}\delta P_{j+1} + \frac{\partial g_j}{\partial r_{j+1}}\delta r_{j+1} \,,
\end{aligned}
\tag{10}
$$

hence we seek to solve

$$
g_j + \frac{\partial g_j}{\partial P_{j-1}}\delta P_{j-1} + \frac{\partial g_j}{\partial r_{j-1}}\delta r_{j-1} + \frac{\partial g_j}{\partial P_j}\delta P_j + \frac{\partial g_j}{\partial r_j}\delta r_j + \frac{\partial g_j}{\partial P_{j+1}}\delta P_{j+1} + \frac{\partial g_j}{\partial r_{j+1}}\delta r_{j+1} \tag{11}
$$

$$
= \mathrm{T} \quad 0 \,.
$$

4 Application

In the general case,

- Equation i calculated at shell j is g_i^j (which is the *"residual"*)
- The derivatives of equation i at shell j with respect to the k stellar structure variables at *the previous shell* $j-1$ are

$$C_{ik}^j = \frac{\partial g_i^j}{\partial x_k^{j-1}} \tag{12}$$

- The derivatives of equation i at shell j with respect to the k stellar structure variables at *the shell* j are

$$D_{ik}^j = \frac{\partial g_i^j}{\partial x_k^j} \tag{13}$$

- The derivatives of equation i at shell j with respect to the k stellar structure variables at *the next shell* $j-1$ are

$$E_{ik}^j = \frac{\partial g_i^j}{\partial x_k^{j+1}} \tag{14}$$

- Each of $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ is an $n \times n$ matrix.
- The equation to be solved is then

$$\begin{pmatrix} \mathbf{D}^1 & \mathbf{E}^1 & & & & & & \\ \mathbf{C}^2 & \mathbf{D}^2 & \mathbf{E}^2 & & & & & \\ & \mathbf{C}^3 & \mathbf{D}^3 & \mathbf{E}^3 & & & & \\ & & & \dots & & & & \\ & & & & \mathbf{C}^{N-2} & \mathbf{D}^{N-2} & \mathbf{E}^{N-2} & \\ & & & & & \mathbf{C}^{N-1} & \mathbf{D}^{N-1} & \mathbf{E}^{N-1} \\ & & & & & & \mathbf{C}^{N} & \mathbf{D}^{N} \end{pmatrix} \begin{pmatrix} \delta\boldsymbol{x}^1 \\ \delta\boldsymbol{x}^2 \\ \delta\boldsymbol{x}^3 \\ \dots \\ \delta\boldsymbol{x}^{N-2} \\ \delta\boldsymbol{x}^{N-1} \\ \delta\boldsymbol{x}^{N} \end{pmatrix} = \begin{pmatrix} \delta\boldsymbol{g}^1 \\ \delta\boldsymbol{g}^2 \\ \delta\boldsymbol{g}^3 \\ \dots \\ \delta\boldsymbol{g}^{N-2} \\ \delta\boldsymbol{g}^{N-1} \\ \delta\boldsymbol{g}^{N} \end{pmatrix}. \tag{15}$$

- This is of the form

$$\boldsymbol{A}\delta\boldsymbol{x} = \boldsymbol{B} \tag{16}$$

i.e. we have to multiply both sides by A^{-1} to find δx,

$$\begin{aligned} A^{-1} A \delta x &= A^{-1} B \\ \delta x &= A^{-1} B \,. \end{aligned} \tag{17}$$

Given δx we have solved the problem.

- The (possibly large) matrix which is to be inverted is then

$$\begin{pmatrix} D^1 & E^1 & & & & & & \\ C^2 & D^2 & E^2 & & & & & \\ & C^3 & D^3 & E^3 & & & & \\ & & & \cdots & & & & \\ & & & & C^{N-2} & D^{N-2} & E^{N-2} & \\ & & & & & C^{N-1} & D^{N-1} & E^{N-1} \\ & & & & & & C^N & D^N \end{pmatrix} . \tag{18}$$

Typically there might be $200 - 2000$ shells: hence $(800 - 8000) \times (4 \times 4 \times 3) \sim 25{,}000 - 250{,}000$ terms in the matrix.

5 The Henyey method

Our Taylor series is, in the more compact notation,

$$-g^j = C^j \delta x^{j-1} + D^j \delta x^j + E^j \delta x^{j+1} \tag{19}$$

where C, D and E are $n \times n$ matrices.

- Assume that the corrections at shell j, δx^j, are linearly related to the corrections at shell $j-1$,

$$\delta x^{j-1} = a^{j-1} + B^{j-1} \delta x^j \,, \tag{20}$$

where a is a vector of size n and B is an $n \times n$ matrix.

- Substitute Eq. 27 into Eq. 19 to find

$$-g^j = C^j \left(a^{j-1} + B^{j-1} \delta x^j \right) + D^j \delta x^j + E^j \delta x^{j+1} \tag{21}$$

$$
\begin{aligned}
&= \mathbf{C}^j \mathbf{a}^{j-1} + \mathbf{C}^j \mathbf{B}^{j-1} \delta\mathbf{x}^j + \mathbf{D}^j \delta\mathbf{x}^j + \mathbf{E}^j \delta\mathbf{x}^{j+1} && (22)\\
&= \mathbf{C}^j \mathbf{a}^{j-1} + \left(\mathbf{C}^j \mathbf{B}^{j-1} + \mathbf{D}^j\right) \delta\mathbf{x}^j + \mathbf{E}^j \delta\mathbf{x}^{j+1} && (23)\\
&= \mathbf{C}^j \mathbf{a}^{j-1} + \mathbf{S}^j \delta\mathbf{x}^j + \mathbf{E}^j \delta\mathbf{x}^{j+1} && (24)
\end{aligned}
$$

which defines

$$
\mathbf{S}^j = \mathbf{C}^j \mathbf{B}^{j-1} + \mathbf{D}^j \, . \tag{25}
$$

Rearranging gives

$$
\mathbf{S}^j \delta\mathbf{x}^j = -\mathbf{g}^j - \mathbf{C}^j \mathbf{a}^{j-1} - \mathbf{E}^j \delta\mathbf{x}^{j+1} \, . \tag{26}
$$

- Repeat for shell $j+1$:

$$
\delta\mathbf{x}^j = \mathbf{a}^j + \mathbf{B}^j \delta\mathbf{x}^{j+1} \, , \tag{27}
$$

pre-mulitply by $\mathbf{S}^j$ and equate to Eq. 26 (at the previous shell),

$$
\begin{aligned}
\mathbf{S}^j \delta\mathbf{x}^j &= \mathbf{S}^j \left(\mathbf{a}^j + \mathbf{B}^j \delta\mathbf{x}^{j+1}\right) && (28)\\
&= -\mathbf{g}^j - \mathbf{C}^j \mathbf{a}^{j-1} - \mathbf{E}^j \delta\mathbf{x}^{j+1} \, . && (29)
\end{aligned}
$$

Now equate terms of the same order

$$
\begin{aligned}
\mathbf{S}^j \mathbf{a}^j &= -\mathbf{g}^j - \mathbf{C}^j \mathbf{a}^{j-1} && (30)\\
\mathbf{S}^j \mathbf{B}^j &= -\mathbf{E}^j \, . && (31)
\end{aligned}
$$

and, multiplying by $\left(\mathbf{S}^j\right)^{-1}$, we find:

$$
\begin{aligned}
\mathbf{a}^j &= -\left(\mathbf{S}^j\right)^{-1} \left[\mathbf{g}^j - \mathbf{C}^j \mathbf{a}^{j-1}\right] \, , && (32)\\
\mathbf{B}^j &= -\left(\mathbf{S}^j\right)^{-1} \mathbf{E}^j \, . && (33)
\end{aligned}
$$

5.1 The first shell

At the first shell ($j = 1$) we have no previous shell on which to depend, hence $C^1 = 0$.

From the definition $S^j = C^j B^{j-1} + D^j$ (Eq. 25) we see

$$S^1 = D^1$$

hence

$$a^1 = -\left(D^1\right)^{-1} g^1 , \qquad (34)$$

$$B^1 = -\left(D^1\right)^{-1} E^1 , \qquad (35)$$

and where D^1, E^1 and g^1 are known (or, at least, can be guessed from previous calculations).

5.2 Subsequent shells

The process is repeated for the rest of the shells. The general expressions for a and B are

$$a^j = -\left(C^j B^{j-1} + D^j\right)^{-1} \left(g^j + C^j a^{j-1}\right) , \qquad (36)$$

$$B^j = -\left(C^j B^{j-1} + D^j\right)^{-1} E^j . \qquad (37)$$

5.3 Final shell

For the N^{th} shell we can write

$$-g^N = C^N \left(a^{N-1} + B^{N-1} \delta x^N\right) + D^N \delta x^N \qquad (38)$$

because $E^N = 0$ (there is no dependence on a non-existent next shell). Hence

$$\delta x^N = -\left(C^N B^{N-1} + D^N\right)^{-1} \left(g^N + C^N a^{N-1}\right) \qquad (39)$$

and finally we have the correction at the end of the matrix.

6 Constructing the solution

If we were clever we saved the $\mathbf{a}$ and $\mathbf{B}$ during the computation, hence from

$$\delta\mathbf{x}^j = \mathbf{a}^j + \mathbf{B}^j \delta\mathbf{x}^{j+1} \quad (40)$$

all the $\delta\mathbf{x}$ are recovered.

The process is iterated until some convergence threshold is satisfied (usually $\delta x_k / x_k < \epsilon$ where $\epsilon \lesssim 10^{-3}$).

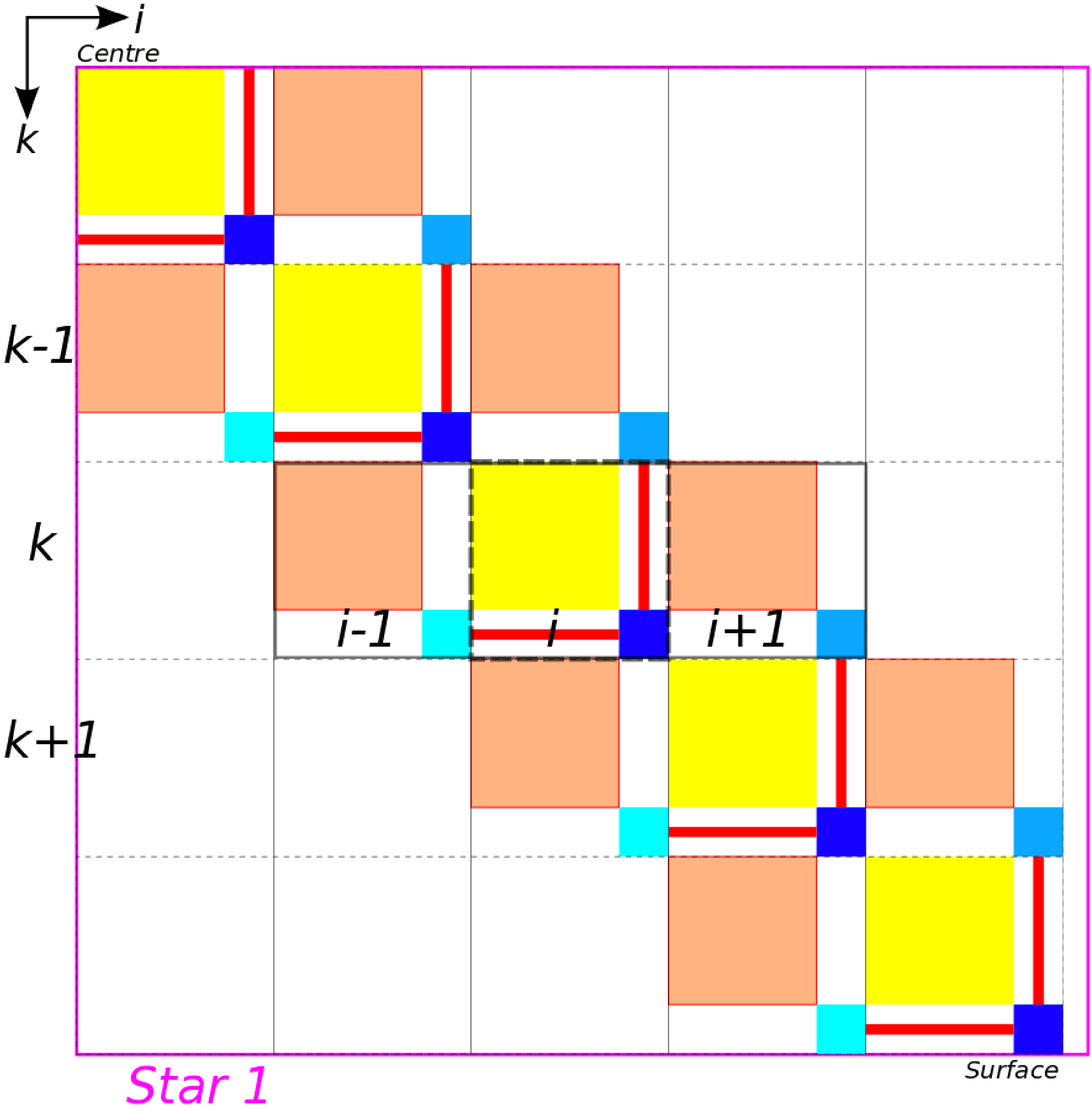

Figure 1: Single star Henyey matrix.

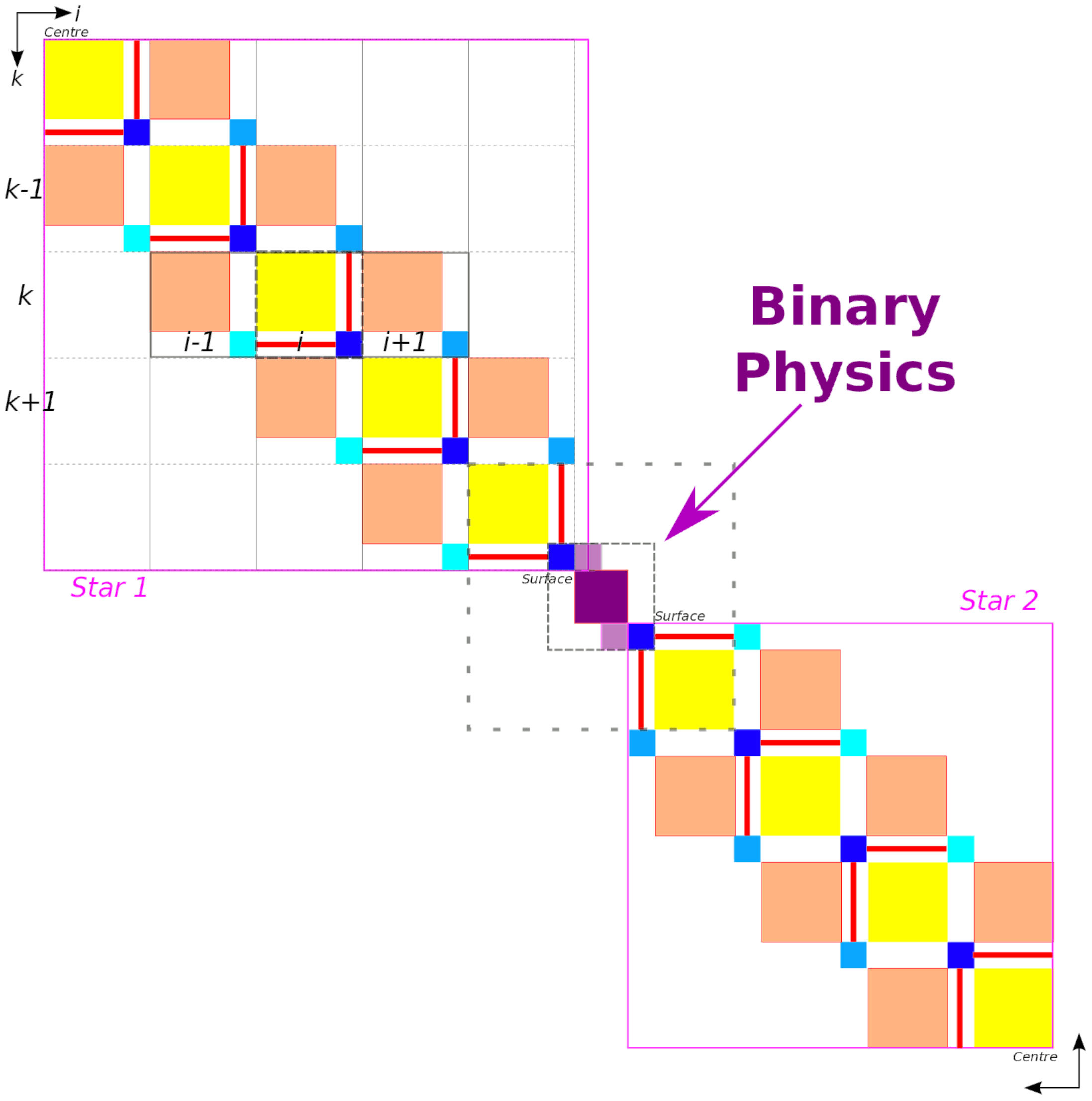

Figure 2: Binary star Henyey matrix.